POLAR RECIPROCAL CONVEX BODIES

BY

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ABSTRACT

The minimum of the product of the volume of a symmetric convex body K and the volume of the polar reciprocal body of K relative to the center of symmetry is attained for the cube and the n-dimensional crossbody. As a consequence, there is a sharp upper bound in Mahler's theorem on successive minima in the geometry of numbers. The difficulties involved in the determination of the minimum for unsymmetric K are discussed.

The product of the volumes of a convex body and its polar reciprocal with respect to an interior point is of importance in the geometry of numbers [6,8], Minkowski geometry [2,3], and differential equations [4].

Let $\mathscr{K}_n$ denote the space of proper convex bodies (compact, convex sets with nonempty interiors) in euclidean n-space E^n with the Blaschke metric, and $\mathscr{S}_n$ the space of symmetric, proper convex bodies. The complete spaces of the convex (respectively, symmetric convex) bodies are denoted by $\mathscr{K}_{\bar{n}}$ and $\mathscr{S}_{\bar{n}}$. For $K \in \mathscr{K}_n$, we denote by K^* the polar reciprocal of K for a point $* \in \operatorname{int} K$, i.e., the body whose distance function for $*$ is the support function of K for the origin $*$. The volume is denoted by V.

By theorems of Blaschke [1] and Santaló [9],

$$\max_{K \in \mathscr{K}_n} \min_{* \in \operatorname{int} K} V(K)V(K^*) = \max_{K \in \mathscr{S}_n} \min_{* \in \operatorname{int} K} V(K)V(K^*) = c_n^2,$$

where c_n is the volume of the unit ball in n-space. Mahler [8,9] has shown

$$\min_{K \in \mathscr{K}_2} \min_{* \in \operatorname{int} K} V(K)V(K^*) = 27/4,$$

$$\min_{K \in \mathscr{S}_2} \min_{* \in \operatorname{int} K} V(K)V(K^*) = 8,$$

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the minima being realized, respectively, by the triangle and the parallelogram. In the note, we generalize Mahler's second result to n-space and prove

$$\min_{K \in \mathscr{S}_n} \min_{* \in \operatorname{int} K} V(K)V(K^*) = 4^n/n!,$$

the minimum being realized by the pair parallelotope-crossbody. The proof follows the general idea of the corresponding proof for locally convex curves with arbitrary winding number [4].

As a consequence, Mahler's fundamental theorem on the successive minima of a symmetric convex body in the lattice of points of integer coordinates (e.g., [8] and [6, Chapter 2, Th. 5]) can be improved to

$$1 \leqq \lambda_i \lambda^*_{n+1-i} \leqq n!$$

where the upper limit is sharp and attained by the polar reciprocal pair parallelotope-crossbody.

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1. The method of proof of Lemma 1 follows an idea of Santaló [10].

LEMMA 1. *The volume $V(K^*)$ is minimal as a function of $* \in \operatorname{int} K$ if and only if $*$ is the center of gravity $g(K^*)$ of K^*.*

Since $V(K)$ does not depend on $*$, the minimum of $f_*(K) = V(K)V(K^*)$ is obtained for $* = g(K^*)$.

Let $p(\omega)$ be the support function of K for unit vectors ω and dS the $(n-1)$-volume element of the unit sphere S^{n-1}. Then,

$$V(K^*) = \frac{1}{n} \int_{S^{n-1}} p^{-n}(\omega) dS(\omega).$$

A change of $*$ by a vector ε induces a change of $p(\omega)$ to

$$\tilde{p}(\omega) = p(\omega) - \varepsilon \cdot \omega.$$

The volume of K^* is changed to

$$V_\varepsilon = V(K^*) + \int_{S^{n-1}} (\varepsilon \cdot \omega) p^{-n-1}(\omega)\, dS(\omega) + \frac{n+1}{2} \int_{S^{n-1}} (\varepsilon \cdot \omega)^2 p^{-n-2}(\omega)\, dS(\omega) + o(|\varepsilon|^2).$$

For $* \in \operatorname{int} K$, $p(\omega) > 0$. The polar equation of the boundary of K^* for the origin

$*$ is $r(\omega) = p^{-1}(\omega)$. It follows that K^*, $V(K_*)$ and $f_*(K)$ are continuous functions of K and $*$, for $* \in \operatorname{int} K$. Since $f_*(K)$ and $V(K^*)$ tend to ∞ as $*$ approaches the boundary of K, the infima of the functions in $\operatorname{int} K$ are also the infima on a compact subset of $\operatorname{int} K$ and are minima.

The point $*$ is stationary for $V(K^*)$ if

$$\int_{S^{n-1}} (\varepsilon \cdot \omega) p^{-n-1}(\omega)\, dS(\omega) = 0 \text{ for all } \varepsilon,$$

and, since the second order term of V_ε is positive, the critical point is a unique minimum. The point $*$ is stationary if and only if the moments of a homogeneous mass distribution throughout K^* vanish for all axes through $*$: the minimum occurs at the centroid of K^*.

The following geometric fact is an immediate consequence of the proof:

COROLLARY. 1. *In the interior of any proper convex body K, there is a unique point $*$ which is the center of gravity of the corresponding polar body K^*.*

2. LEMMA 2. *The function $f(K) = \min_{* \in \operatorname{int} K} f_*(K)$ has a minimum both in $\mathscr{K}_n$ and in $\mathscr{S}_n$.*

The distance and support functions for the origin $*$ are positive, continuous functions of $* \in \operatorname{int} K$. Hence, K^* is a continuous function of $(K, *)$ on an appropriate subspace of $\mathscr{K}_n \times E^n$ and so are $V(K^*)$ and $f_*(K)$. By construction, the function $f(K)$ is lower semicontinuous (in fact, because of $f(K) = f_{g(K^*)}(K)$, $f(K)$ can be shown to be continuous).

There exists a sequence of proper convex bodies K_j such that

$$\lim f(K_j) = \inf_{K \in \mathscr{K}_n} f(K).$$

By a theorem of F. John [5 Th. III], any set M in E^n is contained in some ellipsoid, and a concentric and homothetic image in ratio $1/n$ of that ellipsoid is contained in $\operatorname{conv} M$. This implies that any proper convex body in E^n has an affine image which contains the unit ball of a fixed origin and is contained in the spherical ball of radius n about that origin. Since both the centroid and f_* are affine invariants, f is affinely invariant and we may assume that the K_j contain the unit sphere S^{n-1} about the origin of the coordinates and are contained in nS^{n-1}. By the Blaschke convergence theorem, a subsequence of the K_j converges to a convex set K_0 which is bounded (contained in nS^{n-1}) and proper (containing S^{n-1}). Hence, $K_0 \in \mathscr{K}_n$ and $\lim f(K_j) = f(K_0)$ by the Weierstrass theorem in $\mathscr{K}_n$.

The same argument holds for symmetric bodies; by Lemma 1, $f(K) = f_c(K)$ for a symmetric body K of center c.

3. The following construction is the basis of the proof of the main theorem. Let P be a simplicial polytope of vertices $a_1, \cdots, a_N$. We put the origin at $*$ and identify a_i and the vector $*a_i$. The star $S(a_i)$ of a vertex a_i is the union of all faces of P having a_i as a vertex. For an $(n-1)$-face F_j of $S(a_i)$, let f_j be the vector product of the vectors of the vertices $\neq a_i$ of F_j. (The vector product of $n-1$ vectors $v_1, \cdots, v_{n-1}$ in R^n is the unique vector v for which

$$v \cdot x = \det(v_1, \cdots, v_{n-1}, x)$$

for an arbitrary vector x.) Then,

$$V(\operatorname{conv}(*, S(a_i)) = \frac{1}{n!} a_i \cdot \Sigma f_j, \qquad (F_j \in S(a_i))$$

if the sign of the f_j is derived from the positive orientation of the simplicial complex P derived from that of E^n.

Let P_i be the convex set which is the intersection of the closed halfspaces containing P defined by all $(n-1)$-faces of P that do not belong to $S(a_i)$. For a point x in the intersection of P_i and the support hyperplane π of P of equation

$$(x - a_i) \cdot \Sigma f_j = 0, \qquad (F_j \in S(a_i))$$

we put

$$P' = \operatorname{conv}(a_1, \cdots, a_{i-1}, x, a_{i+1}, \cdots, a_N).$$

Then,

$$V(P) = V(P').$$

In fact, consider P as the union of $C = \operatorname{conv}(*, S(a_i))$ and a remainder set. The polytope P' is the union of the same remainder set and the convex closure C' of x and those $(n-1)$-faces of C that do not contain a_i (i.e., the convex closure of x and the star of $*$ in C). The condition on x assures that $V(C) = V(C')$.

If Q is a crosspolytope of center $*$ and Q' is obtained by replacing two vertices $a_i, -a_i$ by points $x, -x$ according to our construction, then Q' is an affine image of Q in a linear map centered at $*$ and $f_*(Q') = f_*(Q)$.

4. We are now able to prove the main result.

THEOREM. *The minimum of $f(K)$ in $\mathcal{S}_n$ is attained for the crossbody.*

We shall denote by a_i^* the face of the polytope P^* situated in the hyperplane

a_i^*. For $P \in \mathscr{S}_n$, we must consider only the case in which $*$ is the center. The following lemma is true for $\mathscr{S}_n$ but not for $\mathscr{K}_n$.

LEMMA 3. *For every symmetric, simplicial polytope, the central projection of the star of some vertex into the unit sphere of center $*$ is contained in some closed hemisphere.*

The lemma is true for the crossbody; in that case, the central projection of every star is a closed hemisphere.

For every other symmetric, simplicial polytope P, the star of some point a_i must contain more than $2(n-1)$ vertices a_j $(j \neq i)$. Since $S(a_j)$ and $S(-a_j) = -S(a_j)$ cannot intersect in relatively interior points, some $S(a_j)$ must be contained in a closed hemisphere. In fact, $S(a_j)$ is in some closed wedge of angle $< \pi$ whose edge is the line $(a_i, -a_i)$, since otherwise, only $2(n-1)$ such wedges could be accomodated about the axis $(a_i, -a_i)$.

We assume now that the projection of $S(a_i)$ is contained in a closed hemisphere or that $S(a\ )$ is in a closed halfspace H whose boundary contains $*$. By a linear transformation that preserves H, a_i can be mapped into the normal to the boundary of H. The outer normals to the faces of P^* adjacent to $\boldsymbol{a}_i^*$ all point into H. This means that all angles between $\boldsymbol{a}_i^*$ and any adjacent $(n-1)$-face are $\leqq \pi/2$.

In the following, we shall discuss replacing a_i by x. It will be understood that at the same time, we will replace $-a_i$ by $-x$.

i) We assume first that some angles are $< \pi/2$. If $\pi^* \notin \boldsymbol{a}_i$, any point x corresponding to a hyperplane x^* through π^* that separates $*$ from $\boldsymbol{a}_i^*$ generates a P' with $V(P'^*) < V(P^*)$ since $P'^* \subset P^*$. By construction, P' is symmetric simplicial: $f(P)$ is not minimal.

If $\pi^* \in \boldsymbol{a}_i^*$, consider any x^* through π^* and the corresponding symmetric simplicial body P'^*, the intersection of the halfspaces containing $*$ of x^*, $(-x)^*$, and the duals of the remaining vertices of P. If $V(P'^*) < V(P^*)$, $f(P)$ is not minimal. Otherwise, let σ be the reflection in the hyperplane a_i^*. Since $\pi^* \in \sigma x^*$, we have $\sigma x \in \pi$, and σx will be admissable provided that the angle of a_i^* and x^* is sufficiently small. Let P'' be the body defined by σx and $-\sigma x$. Then,

$$\sigma((P^* \setminus P''^*) \cap H) \supset (P'^* \setminus P^*) \cap H,$$

$$\sigma((P^* \setminus P'^*) \cap H) \supset (P''^* \setminus P^*) \cap H.$$

In fact, the two sets involved in any of the inclusions intersect on a_i^*; those of the first inclusion also intersect on x^* and those of the second inclusion on σx^*.

The sets at the right hand sides are also bounded by faces of P^*, the sets at the left hand sides by images under σ of these faces. The inclusions follow since the interior angles that are reflected are $\geqq \pi/2$. By symmetry, similar inclusions hold for $-H$. Since not all angles are $\pi/2$, both inclusions cannot be equalities and $V(P'^*) \geqq V(P^*)$ implies $V(P''^*) < V(P^*)$.

ii) If all angles are $\pi/2$, then P^* is a right prism and P the suspension of an $(n-1)$-dimensional symmetric, simplicial body Q with

$$V(P)V(P^*) = \frac{4}{n} V(Q)V(Q^*),$$

where Q^* is constructed in the boundary hyperplane of H. The previous arguments applied to Q will yield a polytope Q' with $f(Q') < f(Q)$, unless Q is the $(n-2)$-fold, and P the $(n-1)$-fold suspension of a segment, i.e., unless P is a crosspolytope.

In all cases considered here, the number N' of vertices of P' is not greater than the number N of vertices of P. The process can be repeated unless P' is either a crosspolytope or is not simplicial.

Let P be a symmetric polytope that is not simplicial and that is not the union of conv $S(a_i)$ and conv $S(-a_i)$ for all i (i.e., P is not a combinatorial cube.) We choose a pair of vertices $(a_i, -a_i)$ and replace them by $(y, -y)$ where y is close to a_i on the ray $*a_i$ in the exterior of P'. Then $S(y)$ is simplicial in the resulting polytope P'. The construction of Section 3 can be applied to P'. The maximal distance of x from y for which $S(x)$ remains simplicial depends, by Section 3, on the geometry of $P'_i \cap -P'_i$ and also, in a term tending to zero in a continuous way for $y \to a_i$ on $*a_i$, on the position of π. This means that for y close to a_i, there exists $\delta > 0$ for which we may find an x such that for the corresponding P'' we have

$$f(P'') = f_*(P'') < f_*(P') - \delta.$$

In view of the continuity of V and $*$, y can be chosen so that $f_*(P') < f_*(P) + \delta/2$. Hence, $f(P'') < f(P)$ and P'' can be assumed to be simplicial and to have the same number of vertices as P.

Starting from a given simplicial polytope P, we obtain a (possibly transfinite) sequence on which f is strictly decreasing and N is decreasing. Hence, the process must terminate in a crosspolytope Q (or its dual, the parallel body).

Since the simplicial polytopes are dense in all polytopes, there exists a sequence of simplicial polytopes P_j for which

$$\lim f(P_j) = \inf_{P \in \mathscr{S}_n} f(P).$$

By the preceding, $f(P_j) \geqq f(Q)$. Hence, we obtain the assertion of the theorem. In addition, we have seen that the minimum of $f(K)$ on the polytopes is attained only for crossbody-parallel body.

Let $\boldsymbol{P}$ be the metric space of affine equivalence classes of convex bodies in E^n [7]. Then we know that f induces a function $\boldsymbol{f}$ on $\boldsymbol{P}$ that has a unique minimum on the class of crossbodies for all classes of simplicial polytopes. Any convex body K can be approximated arbitrarily closely by simplicial polytopes. If K is not a polytope, the classes of its approximating simplicial polytopes cannot be arbitrarily close to the class of the crosspolytope in $\boldsymbol{P}$ and hence, the lower semicontinuous function $f(K)$ cannot have a minimum for K. As a consequence, we have the complement to the theorem:

The minimum of $f(K)$ in $\mathscr{S}_n$ is attained only for crossbody and parallel body.

5. Finally, we discuss briefly the situation for $\mathscr{K}_n$. The natural conjecture here is that the minimum of $f(K)$ takes place only for the simplex, the simplicial polytope of a minimal number of vertices. The dual of a simplex is a simplex and $* = g(S)$ implies $* = g(S^*)$ for every simplex S.

The proof of Section 4 does not carry over since Lemma 3 does not hold. One therefore has to discuss separately the polytopes for which $* \in \text{int conv } S(a_i)$ for *all* vertices a_i. One shows easily that for polytopes that are the convex closure of the star of one point, this can happen only for the simplex. If $P \neq \text{conv } S(a_i)$ for all i and $n = 3$, then there exist vertices a_j, $a_k \in S(a_i)$ for which $S(a_i) \cap S(a_j) \cap S(a_k)$ consists of at most two points. It follows without difficulty that for $n = 3$, the only polyhedra for which the proof of Section 4 cannot be imitated are the convex closures of three pairwise skew segments. In this case, $f(K)$ can be computed explicitly by a lengthy determinant formula and might be shown to be not minimal. For $n > 3$, the morphology of the exceptional polytopes defies classification and a new approach must be found.

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